TECHNICAL NOTE

Instability in a falling liquid film

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The present note adds to Sha and Soo (1993) concerning the instability of a continuous falling liquid film over a plane, which may lead to the formation of dry patches. Longitudinal wave motion at the surface of a falling liquid film given by Kapitza (1948) has now been extended to include that attributable to transverse disturbances. At a certain average thickness of the film, thin spots of critical film thickness may occur, and the film may split into bands of liquid, thus reducing the effectiveness of cooling.

The coordinates of thin film flow over a cylinder of large diameter are given by: x is the longitudinal coordinate along the wall and in the direction of film flow and gravity, y is parallel to the wall and normal to x, and z is normal to both; u, v, and w are their conjugate components of velocity; the subscript f denotes distributed velocity over the thickness of the film. The momentum integral method, when applied to laminar film flow (Fulford 1964), gives the continuity equation in the following form:

$$w_f|_0^b = -(\partial/\partial x)\int_0^b u_f\,\mathrm{d}z - (\partial/\partial y)\int_0^b v_f\,\mathrm{d}z$$
$$= \mathrm{d}z/\mathrm{d}t\,|_0^b = \mathrm{d}b/\mathrm{d}t\,|_b = \partial b/\partial t + u_s\partial b/\partial x + v_s\partial b/\partial y \quad (1)$$

by integrating for w_f over the height of the film, $z = 0$ to $z = b$, and neglecting evaporation for the present. Here b is the film thickness, and the subscript s denotes the velocity at the surface of the film. The last two terms of Equation 1 were neglected by Kapitza (1948) but are significant when dealing with general wave motion. For laminar motion with a parabolic velocity profile, the surface velocity components of the film are 3/2 of the mean velocity components u and v of the film, or $u_s = (3/2)u$, and $v_s = (3/2)v$, while the integrals of distributed velocity over the thickness of the film b give ub and vb. Equation 1 now becomes:

$$\partial b/\partial t + (3/2)(u\partial b/\partial x + v\partial b/\partial y) = -(\partial ub/\partial x) - (\partial vb/\partial y) \quad (2)$$

The momentum integral equations incorporating the effects of gravity and surface tension in terms of the mean film velocity, now take the form:

$$(\partial u/\partial t) + (9/10)u(\partial u/\partial x) + (6/5)v(\partial v/\partial y)$$
$$-(3/10)u(\partial v/\partial y) - (1/2)(u/b)(\partial b/\partial t)$$
$$-(3/10)(u^2/b)(\partial b/\partial x) - (3/10)(uv/b)(\partial b/\partial y)$$
$$= -(3\nu u/b^2) + g + (\sigma/\rho)(\partial^3 b/\partial x^3) \quad (3)$$

where σ is the surface tension of the liquid, ρ is its density, and

$$(\partial v/\partial t) + (6/5)u(\partial v/\partial x) + (9/10)v(\partial v/\partial y)$$
$$-(3/10)v(\partial u/\partial x) - (1/2)(v/b)(\partial b/\partial t)$$
$$-(3/10)(v^2/b)(\partial b/\partial y) - (3/10)(uv/b)(\partial b/\partial x)$$
$$= -(3\nu v/b^2) + (\sigma/\rho)(\partial^3 b/\partial y^3) \quad (4)$$

The two-dimensional (2-D) disturbances accounted for in Equations 3 and 4 delineate the modifications over that of Kapitza. Some understanding is gained from a solution for small perturbations ('quantities) from steady film flow (subscripts o) (Sha and Soo 1993) by taking: $u = u_o + u'$, $v = v'$, and $b = b_o + b'$. Equations 2 to 4 become, to the first order of primed quantities:

$$-(1/b_o)(\partial b'/\partial t) + (1/2)(u_o/b_o)(\partial b'/\partial x)$$
$$= (\partial u'/\partial x) + (\partial v'/\partial y) \quad (5)$$

$$(\partial u'/\partial t) + (9/10)u_o(\partial u'/\partial x) - (3/10)u_o(\partial v'/\partial y)$$
$$-(1/2)(u_o/b_o)(\partial b'/\partial t) - (3/10)(u_o^2/b_o)(\partial b'/\partial x)$$
$$= -(3\nu u_o/b_o^2)[1 - 2(b'/b_o)] - (3\nu u'/b_o^2) + g$$
$$+(\sigma/\rho)(\partial^3 b'/\partial x^3) \quad (6)$$

$$(\partial v'/\partial t) + (6/5)u_o(\partial v'/\partial x) = -(3\nu v'/b_o^2)$$
$$+(\sigma/\rho)(\partial^3 b'/\partial y^3) \quad (7)$$

Equation 7 shows that the transverse wave is motivated by the principal motion in the x-direction and is a measure of transverse instability. Equations 5–7 can be reduced to one in terms of b' by taking partial derivatives of terms in Equation 6 with respect to x, and those in Equation 7 with respective to y and adding, followed by substitution of Equation 5. One gets a wave equation with b' as the only dependent variable:

$$-(\partial^2 b'/\partial t^2) - (9/10)u_o(\partial^2 b'/\partial t\partial x) + (3/20)u_o^2(\partial^2 b'/\partial x^2)$$
$$= (9\nu u_o/b_o^2)(\partial b'/\partial x) + (3\nu/b_o^2)(\partial b'/\partial t) + (\sigma b_o/\rho)$$
$$\times[(\partial^4 b'/\partial x^4) + (\partial^4 b'/\partial y^4)] \quad (8)$$

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Equation 8 can be solved by postulating b' in the form:

$$b' = b_1 \exp\left[i\left(\omega t - k_x x - k_y y \right) \right] \tag{9}$$

where the wave number $k_y (= 2\pi/\lambda_y$, λ_y is the wave length in the y-direction) corresponds to a transverse perturbation in addition to longitudinal wave number k_x; ω is the circular frequency, and b_1 denotes the amplitude. Substituting into Equation 8, the imaginary (dissipative) component gives:

$$\omega = 3 u_o k_x \quad \text{or} \quad \omega / k_x = 3 u_o = \text{phase velocity} \equiv c \tag{10}$$

the phase velocity, for the principal longitudinal wave motion; a relation obtained in Kapitza (1948) via a different argument. The real part now gives:

$$\omega^2 - (9/10) u_o \omega k_x - (3/20) u_o^2 k_x^2 = (\sigma b_o / \rho)\left(k_x^4 + k_y^4\right) \tag{11}$$

Because u and b are functions of $(x - ct)$ for a given y, the longitudinal motion has the relation:

$$\begin{aligned} & b(c - u_o - u') \\ & \quad = b_o(c - u_o) = b_o\left[1 + (b'/b_o)\right](c - u_o - u') = \text{constant} \end{aligned} \tag{12}$$

giving u'. Substituting Equations 9, 10, and 12 into Equation 6 for a given y with viscous forces balanced by those of gravity gives:

$$k_x = \left(7 u_o^2 \rho / 2 \sigma b_o\right)^{1/2} = 2\pi / \lambda_x \tag{13}$$

(note that 7/2 replaces 4.2 in Kapitza). With the substitution of Equations 10 and 13, Equation 11 now gives $k_y / k_x = (53/70)^{1/4} = 0.93 = \lambda_x / \lambda_y$; i.e., the transverse wave length is slightly larger than that of the longitudinal wave.

The amplitude of fluctuation of u' and b' given by Kapitza (1948) with the rate of energy dissipation by viscosity equal to the rate of change of potential energy by gravity as $b_1 = 0.21 b_o$, and $|u'| \equiv 0.35 u_o$, are now extended to give $|v'| \equiv 0.16 u_o$. Hence, the transverse motion contributes negligible dissipation of energy.

When applied to the film cooling of a nuclear containment vessel (Sha and Soo 1993), the magnitude of the wave lengths of the waves is seen for a water film at room condition as follows:

b_o, mm	0.2	0.4
$b_o - b_1$, mm	0.158	0.316
λ_y, mm	3.25	1.17

$b_o - b_1$, the minimum film thickness at the trough of the waves, can be compared to the critical film thickness given by (Hartley and Murgatroyd 1964) of 0.231 and 0.175 mm for water with a contact angle of 20 and 10° with the surface, respectively. As the film thickness reduces at the trough formed by the waves to its critical value by evaporation, for instance, initiation of dry patches is expected to occur in the above range of flow conditions. The above relation gives a minimum spacing between the liquid streaks.

Important practical modifications to the present theoretical prediction will be the manufacturing tolerance of the surface and its coating, surface contamination, and a possible nonuniformity in the inlet water distribution system in industrial devices.

References

Fulford, G. D. 1964. The flow of liquids in thin films. *Adv. Chem. Eng.*, **5**, 151–236

Hartley, D. E., and Murgatroyd, W. 1964. Criteria for the break-up of thin liquid layers flowing isothermally over solid surfaces. *Int. J. Heat Mass Transfer*, **7**, 1003–1015

Kapitza, P. L. 1948. *Zh. Ekper. Teoret. Fiz.*, **18**, 3 (see Levich, V. G. 1962. *Physicochemical Hydrodynamics*, Prentice-Hall, Englewood Cliffs, NJ, Ch. 12, 670)

Sha, W. T. and Soo, S. L. 1993. Development of steady flow of liquid film over a domed cylinder. *Int. J. Heat Fluid Flow*, **14** 254–259